View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by CERN Document Server

Origin of a classical space in quantum cosmologies

A.A. Kirillov*

G.V. Serebryakov†

Institute for Applied Mathematics and Cybernetics,
10 Ulyanova str., Nizhny Novgorod, 603005, Russia

Abstract

The influence of vector fields on the origin of a classical space in quantum cosmologies and on the possible compactification process in multidimensional gravity is investigated. It is shown that all general features of the transition between classical and quantum regimes of the evolution can be obtained within the simplest Bianchi-I model for arbitrary number of dimensions. It is shown that the classical space appears when the horizon size reaches the smallest of characteristic scales (the characteristic scale of inhomogeneity or the scale associated with vector fields). In multidimensional case the presence of vector fields completely removes the initial stage of the compactification process which takes place in the case of vacuum models [1].

One of the most important problems of quantum cosmology is an adequate description of the origin of the classical background space. Indeed, the transition from the pure quantum regime to the quasiclassical one forms initial properties of the early Universe and, therefore, defines whether subsequent stages contain an inflationary period and which kind of the initial quantum state should be chosen for its realization [2]. In vacuum inhomogeneous models the origin of a classical background space was first investigated in Ref. [3]. It was found that the background metric belongs to the class of quasi-isotropic spaces and the moment of origin of the classical space corresponds to the moment of time when the horizon size matches the scale of inhomogeneity of the space. However, the complete investigation of the problem requires to study the influence of different matter sources and also to reserve the possibility of that our Universe has extra dimensions, as predicted by a number of unified theories [4]. In the last case the transition between quantum and classical regimes should include the so-called compactification stage [5]. And indeed, as it was shown in Refs. [1] the quantum evolution of inhomogeneous models in dimensions less or equal than ten includes

*kirillov@unn.ac.ru

†gvs@focus.nnov.ru

an initial stage of such a compactification. In this case the initial expansion of the Universe goes in an anisotropic way when along extra dimensions scales do decrease and the expansion goes due to only three dimensions.

In this paper we study the influence of vector fields on the origin of a classical space and on the possible compactification process in multidimensional gravity. It turns out that in general, the classical space appears when the horizon size reaches the minimal of scales: the characteristic scale of inhomogeneity or the characteristic scale associated with vector fields which is analog of the Jeans' wave-length (in the present paper we do not consider the inhomogeneous case and therefore we shall discuss the second possibility only). However, in multidimensional case the presence of vector fields completely removes the initial stage of the compactification process which was found in Refs. [1]. This may be a good reason of why vector fields should not be included as external fields but rather should be composed from the additional metric components upon the compactification process.

First, we note that the origin of a background space is not a specific problem for quantum cosmology only. Such a problem does also exist in the classical theory when the gravitational field and matter sources are described by a probabilistic measure distribution with unstable statistical properties. Just this case was shown to be realized in approaching the cosmological singularity Ref.[6]. It turns out that in both cases (in the case of classical or quantum cosmology) the mechanism of the formation of a background space is the same, while the nature of the statistical description is different. Moreover, estimates for the moment of the formation of the background differ by a multiplier which collects all quantum corrections and depends on the choice of initial conditions (and also on the specific scheme of quantization). This ensures the correctness of our consideration and gives the hope that whatever quantum gravity will be constructed in the future, our results will do survive, (save, probably, minor corrections).

Let $A_\mu = (\varphi, A_\alpha)$ be the vector field ($\alpha = 1, 2, \ldots, n$) and for the metric we shall use the standard decomposition

$$ds^2 = N^2 dt^2 - g_{\alpha\beta}\left(dx^\alpha + N^\alpha dt\right)\left(dx^\beta + N^\beta dt\right). \tag{1}$$

Then the action takes the form (in what follows we use the Planckian units)

$$I = \int d^n x dt \left\{ \pi^{\alpha\beta} \frac{\partial}{\partial t} g_{\alpha\beta} + \pi^\alpha \frac{\partial}{\partial t} A_\alpha + \varphi \partial_\alpha \pi^\alpha - N H^0 - N^\alpha H_\alpha \right\}, \tag{2}$$

where

$$H^0 = \frac{1}{\sqrt{g}} \left\{ \pi^\alpha_\beta \pi^\beta_\alpha - \frac{1}{n-1} \left(\pi^\alpha_\alpha\right)^2 + \frac{1}{2} g_{\alpha\beta} \pi^\alpha \pi^\beta + V \right\}, \tag{3}$$

$$H_\alpha = -\nabla_\beta \pi^\beta_\alpha + \pi^\beta F_{\alpha\beta}, \tag{4}$$

here $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, $V = g\left(\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - R\right)$and R is the scalar curvature with the metric $g_{\alpha\beta}$. Variating the action with respect to φ we find the constraint

$\partial_\alpha \pi^\alpha = 0$, so it suffices to consider only transverse parts for A_α and π^α. Thus in what follows we set $\varphi = 0$ and $\partial_\alpha \pi^\alpha = 0$ to be satisfied.

It is convenient to use the so-called generalized Kasner-like parametrization of the dynamical variables [6, 1]. The metric components and their conjugate momenta are represented as follows

$$g_{\alpha\beta} = \sum_a \exp\{q^a\}\, \ell_\alpha^a \ell_\beta^a\,, \qquad \pi_\beta^\alpha = \sum_a p_a L_a^\alpha \ell_\beta^a\,, \tag{5}$$

where $L_a^\alpha \ell_\alpha^b = \delta_a^b$ $(a, b = 0, ..., (n-1))$, and the vectors ℓ_α^a contain only $n(n-1)$ arbitrary functions of spatial coordinates. Further parametrization may be taken in the form

$$\ell_\alpha^a = U_b^a S_\alpha^b, \qquad U_b^a \in SO(n), \qquad S_\alpha^a = \delta_\alpha^a + R_\alpha^a \tag{6}$$

where R_α^a denotes a triangle matrix ($R_\alpha^a = 0$ as $a < \alpha$). Substituting Eq.(5), (6) into (2) we find the following expression for the action functional

$$I = \int_S (p_a \frac{\partial q^a}{\partial t} + T_a^\alpha \frac{\partial R_\alpha^a}{\partial t} + \pi^\alpha \frac{\partial A_\alpha}{\partial t} - N H^0 - N_\alpha H^\alpha) d^n x dt, \tag{7}$$

where $T_a^\alpha = 2\sum_b p_b L_b^\alpha U_a^b$ and the Hamiltonian constraint acquires the structure

$$H^0 = \frac{1}{\sqrt{g}} \left\{ \sum p_a^2 - \frac{1}{n-1} (\sum p_a)^2 + \frac{1}{2} \sum e^{q^a} (\pi^a)^2 + V \right\}. \tag{8}$$

In the last equation the potential V collects all spatial derivatives and we used the notion $\pi^a = \sum \pi^\alpha \ell_\alpha^a$. In the case of $n = 3$ the functions R_α^a are connected purely with transformations of a coordinate system and may be removed by resolving momentum constraints $H^\alpha = 0$ [6]. However, in the multidimensional case the functions R_α^a contain $\frac{n(n-3)}{2}$ dynamical functions as well.

It can be shown that near the singularity in the case $n > 3$ all spatial derivatives can be neglected in the leading order. Therefore, we neglect the potential V (terms $F_{\alpha\beta}$ and R) in the action (8). In the case $n = 3$, the curvature term cannot be neglected. However, in this case the kinetic term of the vector field ($\frac{1}{2} g_{\alpha\beta} \pi^\alpha \pi^\beta$) induces exactly the same type of the evolution (from qualitative and even quantitative viewpoint) as that of the curvature terms.

Thus, in this approximation Einstein equations formally coincide with equations for homogeneous Bianchi-I model and it is a remarkable fact that near the singularity homogeneous Bianchi -I model with a vector field contains all qualitative features of the general inhomogeneous models. In what follows we shall use the gauge $N^\alpha = 0$ and for the sake of simplicity consider homogeneous model, i.e., all functions to depend only on time. We also use the normalization of the space volume $V^n = \int_S d^n x = 1$. Thus, we find (within our approximation) equations for the vector field

$$E_\alpha = \frac{\partial}{\partial t} A_\alpha = \frac{N}{\sqrt{g}} g_{\alpha\beta} \pi^\alpha, \tag{9}$$

$$\frac{\partial}{\partial t}\pi^{\alpha}=0. \qquad (10)$$

The last equation gives $\pi^{\alpha}=const$ and in what follows we shall suppose values π^{α} as external parameters (which actually are eigenstates of the respective operators). The rest of the present paper repeats mainly the method suggested in Ref. [3]. Near the singularity it is convenient to make use of the following parametrization of the scale functions [6]

$$q^{a}=\ln R^{2}+Q_{a}\ln g;\ \sum Q_{a}=1, \qquad (11)$$

where we distinguished a slow function of time R which characterizes the absolute value of the metric functions [7, 8] and is specified by initial conditions (see below) and the anisotropy parameters Q_q and $\ln g=\sum q^{a}-6\ln R$ can be expressed in terms of a new set of variables τ, y^{i} $(i=1,2...(n-1))$, as follows

$$Q_{a}\left(y\right)=\frac{1}{n}\left(1+\frac{2y^{i}A_{i}^{a}}{1+y^{2}}\right),\quad \ln g=-ne^{-\tau}\frac{1+y^{2}}{1-y^{2}} \qquad (12)$$

where A_{i}^{a} is a constant matrix, e.g., see in Refs. [6]. The parametrization (12) has the range $y^{2}<1$ and $-\infty<\tau<\infty$, $(0\leq g\leq 1)$ and an appropriate choice of the function R allows to cover, by this parametrization, all of the classically allowed region of the configuration space.

The evolution (rotation) of Kasner vectors results in a slow dependence of functions π^{a} on time and it can be shown that these functions are completely determined by the momentum constraints, while the evolution of scale functions is described by the action

$$I=\int\{(\vec{P}\frac{\partial\vec{y}}{\partial t}+h\frac{\partial\tau}{\partial t})-\frac{N}{n\left(n-1\right)R^{3}\sqrt{g}}e^{2\tau}[\varepsilon^{2}-h^{2}+U(\tau,\vec{y})]\}dt, \qquad (13)$$

where $\varepsilon^{2}=\frac{1}{4}\left(1-y^{2}\right)^{2}\vec{P}^{2}$ and the potential term U (which comes from the kinetic term for the vector field) has the following structure

$$U=n\left(n-1\right)R^{2}e^{-2\tau}\sum_{a=1}^{n}\left(\pi^{a}\right)^{2}g^{Q_{a}}. \qquad (14)$$

Here the coefficients π^{a} (projections of π^{α} on Kasner vectors) are slow functions of $\ln g$ and characterize an initial intensity of the vector field. In the approximation of deep oscillations, when $g\ll 1$, this potential can be modeled by a set of potential walls

$$g^{Q_{a}}\to\theta_{\infty}[Q_{a}]=\begin{cases}+\infty\ , & Q_{a}<0,\\ 0\ , & Q_{a}>0,\end{cases} \qquad (15)$$

and is independent of Kasner vectors $U_{\infty}=\sum\theta_{\infty}\left(Q_{a}\right)$.

By solving the Hamiltonian constraint $H = 0$ in (13) we define the ADM action [9] reduced to the physical sector as follows

$$I = \int (\vec{P}_{\vec{y}} \cdot \frac{d\vec{y}}{d\tau} - H_{ADM}) d\tau, \tag{16}$$

where $H_{ADM} \equiv -h = \sqrt{\varepsilon^2 + U}$ is the ADM Hamiltonian and τ plays the role of time ($\dot{\tau} = 1$) which corresponds to the gauge $N_{ADM} = \frac{n(n-1)R^3\sqrt{g}}{2H_{ADM}} e^{-2\tau}$.

The condition of applicability of the approximation (15) can be written as follows

$$\varepsilon^2 \gg U \tag{17}$$

as $Q_a > \delta > 0$ ($\delta \ll 1$). Thus, from the condition that the approximation of deep oscillations (15) breaks at the moment $g \sim 1$, one finds that the function R should be chosen as follows $R^2 = \frac{\varepsilon^2}{n(n-1)\pi^2} e^{2\tau}$ (where $\pi^2 = \sum (\pi^a)^2$) and the inequality (17) reads $g \ll 1$.

The synchronous cosmological time relates to τ by means of the equation $dt = N_{ADM} d\tau$ from which we find the estimate $\sqrt{g} \sim t/t_0$, where $t_0 = cL^3\varepsilon^2$,(compare with Ref. [3]) $L \sim 1/\pi$ is a characteristic scale related to the vector field, ε is the ADM energy density ($\varepsilon = const$), and c is a slow (logarithmic) function of time ($c \sim 1$ as $g \to 1$). Thus, in the synchronous time the upper limit of the approximation (15) is $t \sim t_0$. We note that from the physical viewpoint t_0 corresponds to the moment when the horizon size reaches the characteristic scale related to the energy of vector field and both terms in the Hamiltonian constraint (the kinetic energies of the anisotropy and of the vector field) acquire the same order.

The physical sector of the configuration space (variables $\vec{y}$) is a realization of the Lobachevsky space and the potential U_∞ limits the part $K = \{Q_a \geq 0\}$. Quantization of this system can be carried out as follows. The ADM density of energy represents a constant of motion and, therefore, we can define stationary states as solutions to the eigenvalue problem for the Laplace - Beltrami operator $-\varepsilon^2 = \Delta + \frac{(n-2)^2}{4} P$ (see Refs. [10, 1])

$$(\Delta + k_n^2 + \frac{(n-2)^2}{4} P)\varphi_n(y) = 0, \qquad \varphi_n \mid_{\partial K} = 0, \tag{18}$$

where the Laplace operator Δ is constructed via the metric $\delta l^2 = h_{ij} \delta y^i \delta y^j = \frac{4(\delta y)^2}{(1-y^2)^2}$. The eigenstates φ_n are classified by the integer number n and obey the orthogonality and normalization relations

$$(\varphi_n, \varphi_m) = \int_K \varphi_n^*(y)\, \varphi_m(y) D\mu(y) = \delta_{nm}, \tag{19}$$

where $D\mu(y) = \frac{1}{a_n} \sqrt{h} d^2 y\,(x)$ and a_n is the volume of K. Thus, an arbitrary solution Ψ to the Shredinger equation $i\partial_\tau \Psi = H_{ADM}\Psi$ takes the form

$$\Psi = \sum_n \exp(-ik_n\tau)\varphi_n(y)C_n \tag{20}$$

where C_n are arbitrary constants which are to be specified by initial conditions. Within our approximation these constants also represent arbitrary functions of the vector field

$$C_n(A) = \int d^{n-1}\pi C_{n,\pi} \exp\left(i\pi^\alpha A_\alpha\right) \tag{21}$$

and the normalization condition reads ($\int d^{n-1}A \sum_n \mid C_n(A) \mid^2 = 1$). The probabilistic distribution for variables y has the standard form $P(y,\tau) = |\Psi(y,\tau)|^2$. The eigenstates φ_n define stationary (in terms of the anisotropy parameters $Q(y)$) quantum states and describe an expanding universe with a fixed energy of the anisotropy.

For an arbitrary quantum state Ψ we can determine the background metric $\left\langle ds^2 \right\rangle$. However such a background is stable and has sense only when quantum fluctuations around it are small. In the case of $g \ll 1$ fluctuations well exceed the average metric and the background is hidden [10]. Indeed, consider the moments of scale functions $\left\langle a_i^M \right\rangle = \left\langle R^M g^{\frac{M}{2}Q_i} \right\rangle$. The leading contribution in $\left\langle a_i^M \right\rangle$ comes from those points of the billiard at which Q_i takes the minimal value. Such points are at the boundary of the billiard and the minimal value of Q is $Q_i^{\min} = 0$. Since $\varphi_J(\partial K) = 0$, in the neighborhood of ∂K we have $\varphi_J \approx \eta_J Q$ and the probability density is

$$P_\tau(Q) = \int_K P(y,\tau)\,\delta(Q - Q(y))\sqrt{h} d^{n-1}y \approx B_m(\tau)\, Q^{f(n)} \tag{22}$$

as $Q \to 0$. Here $f(n) = 2$ for $n > 3$ and $f(3) = 3/2$ (since in the case $n = 3$ we get $\sqrt{h} \sim 1/\sqrt{Q}$). Thus, in the limit $g \to 0$ moments of the function $\left\langle a_i^M \right\rangle$ are given by ($M > 0$)

$$\left\langle a_i^M \right\rangle \simeq D_i(M,\tau)\frac{1}{(M \ln 1/g_*)^{f(n)+1}}, \tag{23}$$

where $g_* = g(\tau, y^*)$ and D_i is a slowly varying in time function which collects information of the initial quantum state.

Consider now an arbitrary stationary state φ_n which gives the stationary probabilistic distribution $P(y) = |\varphi_n|^2$. In this case $D \simeq b{k_n}^M \left(L^M\right)_n$ is a constant, where b comes from the uncertainty in the operator ordering. Thus, for the intensity of quantum fluctuations one finds the divergent, in the limit $g \to 0$ ($\tau \to -\infty$), expression $\left\langle \delta^2 \right\rangle = \left(\left\langle a^2 \right\rangle / \left\langle a \right\rangle^2 - 1\right) \sim (\ln 1/g_*)^{f(n)+1}$ which explicitly shows the instability of the average geometry as $g \ll 1$. The intensity of quantum fluctuations reaches the order $\delta \sim 1$ at the moment $t \sim \left(L^3\right)_n k_n^2$

($g \sim 1$) when the anisotropy functions can be described by small perturbations $a_i^2 = R^2 \left(1 + Q_i \ln g + \ldots\right)$ and the universe acquires a quasi-isotropic character. This moment can be considered as the moment of the origin of a stable classical background. It is important that a strong anisotropy for a stable classical geometry turns out to be forbidden. We recall that the hypothesis that the very beginning of the evolution of the universe should be described by a quasi-isotropic models, while anisotropic models are forbidden was first suggested in Ref. [11] from a different consideration (as a result of the imposibility to construct a self-consistent theory which could account for the back reaction of the particle creation in such models).

To conclude we make two important remarks. Firstly, in the presence of the vector field the evolution of the metric undergoes spontaneous stochastization for arbitrary number of dimensions [12]. This follows from the fact that potential walls always restrict a finite region of the configuration space ($Vol.\,(K) < \infty$ for arbitrary n) and the metric should be described by an invariant measure. In this case estimates for moments of scale functions can be obtained by setting $P\,(y,\tau) = 1$ which gives the replacement $f(n) \to f(n) - 2$ in (23). Thus the above picture of the formation of a stable background remains to be valid in the classical theory as well (with minor corrections of estimates). And secondly, the presence of the vector field drastically changes the structure of the configuration space in dimensions $n > 3$. The billiard turns out to be finite for an arbitrary number of dimensions [12] but the payment is the fact that the initial stage of the compactification process which was found in vacuum models [1] is absent. Indeed, in the model considered above the anisotropy parameters have the range $1 \geq Q \geq 0$ and remain always to be positive (we recall that in the vacuum case these functions could have negative values, e.g., they changed in the range $1 \geq Q \geq -\,(n-3)\,/\,(n+1)$). This means that the expansion always goes in such a way that lengths monotonicaly increase in all spatial directions. This probably represents a rather serious reason of why vector fields should not be included as external fields in multidimensional gravity.

This research was supported by RFBR (Grant No. 98-02-16273) and DFG (Grant No. 436 Rus 113/23618).

References

[1] A.A. Kirillov, Pis'ma Zh. Eksp. Teor. Fiz. **62** , 81 (1995) [JETP Lett. **62**, 89 (1995)].

[2] A.D. Linde, Particle Physics and Inflationary Cosmology (Harwood Academic, 1990).

[3] A.A. Kirillov and G. Montani, Pis'ma Zh. Eksp. Teor. Fiz. **66**, 449 (1997).

[4] M.B. Green, J.H. Schwarz, and E. Witten, *Superstring Theory*, Cambridge Univ. Press, 1988.

[5] A. Chodos and S. Detweiler, Phys. Rev., **D21**, 2176 (1980); M. Demianski, Z. Golda, M. Heller and M. Szydlowski, Class. Quantum Grav. **3**, 1196 (1986); *ibid.*, **5**, 733 (1988); K. Maeda and P.Y. Pang, Phys. Lett., **180B**, 29 (1986).

[6] A.A. Kirillov, Zh. Eksp. Teor. Fiz. **103**, 721 (1993). [Sov. Phys. JETP **76**, 355 (1993)]; A.A. Kirillov and V.N. Melnikov, Phys.Rev. **D51** 723 (1995).

[7] C.W. Misner, Phys. Rev. Lett. **22**, 1071 (1969).

[8] C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), Vol.2

[9] R. Arnowitt, S. Deser, and C.W. Misner, In Gravitation: *An Introduction to Current Research*, edited by L.Witten, Wiley, New York, 1962 ,P.227.

[10] A.A. Kirillov, Phys. Lett. **B 399**, 201 (1997).

[11] V. N. Lukash, A.A. Starobinsky, Zh. Eksp. Teor. Fiz. **66**, 1515 (1974).

[12] T.Damour and M.Henneaux, Phys. Rev. Lett. **85** 3761 (2000).